Numerical Methods

NON-LINEAR PROBLEMS

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AN ERROR MINIMIZING SCHEME FOR THE NON-LINEAR SHALLOW-WATER WAVE EQUATIONS WITH MOVING BOUNDARIES

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INTRODUCTION

A numerical method is presented for the solution of the shallow-water wave equations on time dependent domains. Particular interest is given to the modeling of transient irregularly shaped boundaries. This problem occurs, e.g. in the numerical simulation of storm surges and in the calculation of flow fields in tidal rivers and estuaries.

The method discussed is a special case of the adaptive (or moving) grid technique [5]. However, the movement of grid points is permitted only at the boundary, while interior grid points remain fixed. Hence, it is possible to combine the moving boundary technique with other Finite Difference/Finite Element Methods on a fixed, regular grid.

The moving boundary is part of the solution. The customary procedures either solve an implicit equation for the boundary points or make assumptions of an explicit dependence of the boundary points on, for example, the flow field (e.g. [6]). In the method employed in this paper the equations of the boundary points are derived directly from the shallow-water equations.

This method is advantageous in cases of internal discontinuities, i.e. steep gradients, interfaces, shocks, rare faction waves etc.. In practice it is difficult to numerically calculate the positions of these discontinuities since their propagation velocity is not known a priori. Therefore, internal boundary conditions are added to the differential equation [1]. It should be mentioned that the use of an overall adaptive grid will enhance greatly the accuracy if e.g., bores appear in the solution. One has to keep in mind, however, that the shallow-water equation is not applicable to wave-breaking problems, so only approximate results can be expected.

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1. THE BASIC EQUATIONS

From the Euler-equations and the continuity equation where an incompressible fluid is assumed, the following formulas are obtained by integration over the depth according to shallow-water wave theory.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial h}{\partial x} =: \frac{\partial u}{\partial t} + L_u(u,h) = 0$$
$$\frac{\partial h}{\partial t} + \frac{\partial Du}{\partial x} =: \frac{\partial h}{\partial t} + L_h(u,h) = 0 \qquad (1.1)$$

where u is the average velocity, h denotes free surface elevation with respect to the still water level and D is the total water depth (D = h+H).

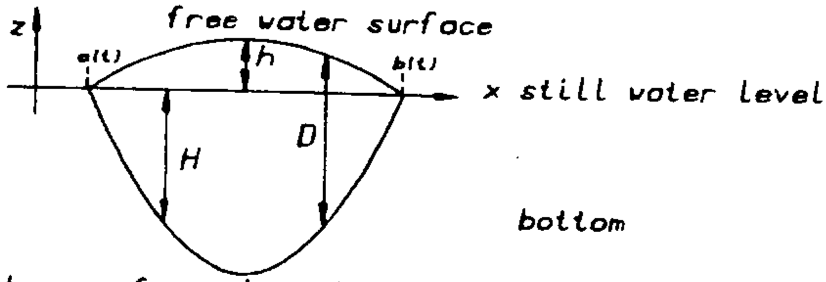

Fig.1: Coordinate system

Introducing typical quantities such as a time scale T, an average depth $\bar{H}$ and a length $L = \sqrt{g\bar{H}}\,T$, the following dimensionless variables can be defined:

$$x \to \frac{x}{L},\quad D \to \frac{D}{\bar{H}},\quad h \to \frac{h}{\bar{H}},\quad t \to \frac{t}{T},\quad u \to u\frac{T}{L}\,.$$

To enhance computational stability of the continuity equation, which describes surface elevation h, a numerical viscosity term uh_{xx} can be added [4].

2. THE VARIATIONAL PRINCIPLE FOR A MOVING COORDINATE SYSTEM

A moving coordinate system is described by a strictly monotone, continuous grid function $x = x(\xi,t)$ on the fixed parameter space $P = [0, N+1] \times \mathbb{R}^+ \to \mathbb{R}$ (in practice P is mapped onto $\Omega(t)$, see below) where the grid points are simply the function values $x_i^t := x(i,t)$, $i = 0,\ldots, N+1$, and ξ denotes positions in the parameter space P. Now the solution functions $u = u(x(\xi,t),t)$, $h = h(x(\xi,t),t)$, which are in $C^1[\Omega(t) \times \mathbb{R}^+]$, where $\Omega(t) = [a(t), b(t)]$ is the time dependent physical domain (see Fig. 1), can be transformed to the fixed parameter space P by means of the grid function $x = x(\xi,t)$.

With the choice of the transformed trial functions u,h out of the space $X_1 \times C^1(\mathbb{R}^+)$, the grid function $x \in X_2 \times C^1(\mathbb{R}^+)$ where $X_i \subset C^0 [0, N+1]$, i = 1,2, finite dimensional spaces, i.e. u,h are piecewise polynomials, the differential equation is not exactly satisfied. As grid generation principle one chooses the minimum of the error which is defined as follows:

$$\alpha \left\| \frac{\partial u}{\partial t} + L_u(u,h) \right\|_2^2 + (1-\alpha) \left\| \frac{\partial h}{\partial t} + L_h(u,h) \right\|_2^2 \rightarrow \text{minimum.} \quad (2.1)$$

Here $\alpha \in [0,1]$ is a weight factor, L_u, L_h contain the spatial derivatives in the system (1.1), $\| \ \|_2$ is the norm in the Hilbert space $L_2(\Omega(t))$. Minimization is performed with respect to the time derivatives $\dot{u}, \dot{h}, \dot{x}$ of the unknown functions in the parameter space such that the boundary conditions are satisfied. This principle is consistent with the hyperbolic character of equations (1.1). The temporal behavior of the numerical solutions of u,h and x is calculated in an optimal fashion (see Eq. (2.1) depending on the specified initial values at time instant t_n.

For the calculation of the total derivative with respect to t, we find

$$\dot{u} := \frac{du(x,t)}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{\partial x}{\partial t}$$

$$\frac{\partial u}{\partial t} = \dot{u} - u_x \dot{x}, \quad \frac{\partial h}{\partial t} = \dot{h} - h_x \dot{x}$$

where $x = x(\xi,t)$ was used.

The variational equations for the minimization problem follow from Eq. (2.1) by differentiation with respect to $\dot{u}, \dot{h}$ and $\dot{x}$ resulting in

$$\int_{\Omega(t)} (\dot{u} - u_x\dot{x} + L_u(u,h))\phi_1^i \, dx = 0, \quad \text{for } i = 1 \text{ to } \dim X_1 \quad (2.2)$$

$$\int_{\Omega(t)} (\dot{h} - h_x\dot{x} + L_h(u,h))\phi_1^i \, dx = 0, \quad \text{for } i = 1 \text{ to } \dim X_1 \quad (2.3)$$

$$\alpha \int_{\Omega(t)} (\dot{u} - u_x\dot{x} + L_u(u,h)) (-u_x\phi_2^i) dx + \quad (2.4)$$

$$(1-\alpha) \int_{\Omega(t)} (\dot{h} - h_x\dot{x} + L_h(u,h)) (-h_x\phi_2^i) dx = 0, \quad \text{for } i = 1 \text{ to } \dim X_2$$

where ϕ_1^i and ϕ_2^i form finite sets of basis functions for the spaces X_1 and X_2, respectively.

$$u_x(x,t) = \sum_{i=0}^{N+1} u_i(t) \frac{\partial}{\partial x} \phi_i^t(x) = \frac{u_{i+1}(t) - u_i(t)}{x_{i+1}(t) - x_i(t)} \quad \text{for } x \in [x_i, x_{i+1}]$$

The expressions for $\dot{D}$, D_x and $\dot{x}$ are obtained in the same manner. These derivatives are inserted into the variational equations (2.2 - 2.4). The integration over $\Omega(t)$ is reduced to the intervall $[x_{i-1}, x_{i+1}]$ since the basis functions ϕ_i^t vanish outside.

With the solution vector $z := \begin{pmatrix} u_i \\ D_i \\ x_i \end{pmatrix}_{i=0,\ldots,N+1}$ a system of nonlinear, weakly coupled differential equations in time is obtained, which is solved by either an explicit fourth order Runge-Kutta scheme or by Gear's implicit method. The left-hand side of Eq. (3.1) results from combining the time derivatives of Eqs. (2.2 - 2.4) whereas the spatial derivatives give rise to the vector L, described below. The differential equations are formally written as

$$A(z)\dot{z} = L(z) \tag{3.1}$$

$$\text{where } A(z) = \begin{pmatrix} A_0 & B_0 & & & & \\ C_1 & \ddots & \ddots & & 0 & \\ & \ddots & \ddots & \ddots & & \\ & & C_i & A_i & B_i & \\ & 0 & & \ddots & \ddots & B_N \\ & & & & C_{N+1} & A_{N+1} \end{pmatrix},$$

and $A_i = D_{ii}$, $B_i = D_{i,i+1}$, $C_i = D_{i-1,i}$

with the elementary matrices

$$D_{ij} = \begin{pmatrix} \langle\phi_i^t, \phi_j^t\rangle, & 0, & -\langle u_x\phi_i^t, \phi_j^t\rangle \\ 0, & \langle\phi_i^t, \phi_j^t\rangle, & -\langle D_x\phi_i^t, \phi_j^t\rangle \\ -\alpha\langle\phi_i^t, u_x\phi_j^t\rangle, & -(1-\alpha)\langle\phi_i^t, D_x\phi_j^t\rangle, & \alpha\langle u_x\phi_i^t, u_x\phi_j^t\rangle + (1-\alpha)\langle D_x\phi_i^t, D_x\phi_j^t\rangle \end{pmatrix}$$

where $\langle f,g\rangle = \int_{\Omega(t)} fg\,dx$ is the scalar product in $L_2(\Omega(t))$

and the right-hand side

$$L(z) = \begin{pmatrix} -\langle L_u(u,D), \phi_i^t\rangle \\ -\langle L_h(u,D), \phi_i^t\rangle \\ \alpha\langle L_u, u_x\phi_i^t\rangle + (1-\alpha)\langle L_h, D_x\phi_i^t\rangle \end{pmatrix}_{i=0,\ldots,N+1.}$$

One observes here a feature of the grid generation method, which we call a duality principle:

> If for any integer i both functions $u_x \phi_2^i$ and $h_x \phi_2^i$ can be expressed as a linear combination of basis functions ϕ_1^i, the system of Eqs. (2.2 - 2.4) becomes linear dependent. Hence, function $x(\xi,t)$ cannot be uniquely determined from this system, that is grid point positions are undetermined.

Let $X_1 = X_2 = C^1[0, N+1]$. Then u,h calculated by Eqs. (2.2 - 2.4) satisfy the exact equations (1.1) regardless of the choice of the grid function.

3. FINITE ELEMENT FORMULATION FOR SHALLOW-WATER EQUATIONS

For the actual computations we replace in Eqs. (1.1) the surface elevation h by the total water depth D as the new independent variable. Then we choose

$$X_1 = X_2 = X = \{\phi \in C^0[0, N+1],\ \phi \text{ piecewise linear in } [i,i+1],\ i = 0, \ldots, N\}.$$

Because of this choice it is not necessary to discern between basis functions ϕ_1^i and ϕ_2^i and therefore basis functions are denoted by ϕ_i from hereon. Furthermore it must be guaranteed that x is monotone (see section 1) in order to prevent crossing of grid points. This property is generally established by the use of regularisation terms [5]. However, this problem is of no concern if only the movement of the boundary points is considered. Then x is invertible, that is $\xi = x^{-1}(x,t)$, hence one can write

$$x(\xi,t) = \sum_{i=0}^{N+1} x_i(t)\phi_i(\xi)$$

$$u(x,t) = \sum_{i=0}^{N+1} u_i(t)\phi_i(x^{-1}(x,t)) =: \sum_{i=0}^{N+1} u_i(t)\phi_i^t(x)$$

$$D(x,t) = \sum_{i=0}^{N+1} D_i(t)\phi_i^t(x)$$

where $x_i(t)$, $u_i(t)$ and $D_i(t)$ are the time dependent coefficients which have to be determined from Eqs. (2.2 - 2.4) and ϕ_i is the usual piecewise linear finite element basis function at node $\xi = i$.

From the above equations one gets the spatial and temporal derivatives

$$\dot{u}(x,t) = \sum_{i=0}^{N+1} \dot{u}_i(t)\phi_i^t(x)$$

The duality principle appears now in the form:

The fact that $u_x \phi_i^t, D_x \phi_i^t \in X$ (i.e. are continuous and piecewise linear) is equivalent to u_x, h_x = constant for $x \in \text{supp}\, \phi_i^t$ (i.e. $[x_{j-1}, x_{j+1}]$). That means: If u and D are straight lines passing through the node x_j, then the equation for x_j will be linearly dependent on those for u and D, so that x_j cannot be calculated.

4. BOUNDARY CONDITIONS

The characteristics of the shallow-water equations are given by the eigen-values $\lambda_{1,2} = u \pm \sqrt{gD}$ which follow from the two equations (1.1).
Hence $\frac{dx}{dt} = u \pm \sqrt{gD}$ holds along a characteristic.

Each characteristic will reach the moving boundary $\partial\Omega(t)$ after a finite time. Correspondingly, from each boundary point exactly one characteristic starts into the interior. This holds, because the distance $s = |x - x_b|$ between a characteristic and the boundary point x_b obeys the differential equation

$$\frac{ds}{dt} = u(x) \pm \sqrt{gD(x)} - u(x_b),$$

which follows from the difference of the velocities at the boundary point and the current position x.
Expansion of u(x) and D(x) at x_b leads to

$$\frac{ds}{dt} = \pm \sqrt{g|D_x|}\, s^{1/2} + O(s).$$

Solving for s results in

$$s = \left(x - \frac{\sqrt{g|D_x|}}{2}\, t \right)^2$$

which is valid only in the vicinity of the boundary.
This formula shows, since $|D_x(x_b)|$ is different from zero, that a characteristic starting from a position x, reaches the boundary in a finite time. Moreover, characteristics radiating from different positions at the same instant of time cannot intersect at the boundary. Hence the boundary $\partial\Omega(t)$ is not included in the domain of dependence of the initial distribution (for $t > 0$).

Therefore, exactly one boundary condition for either u or h has to be specified. In the case of a moving boundary this is obviously the condition $D = 0$, because the boundary is defined by this equation.

$$= u_1 + \frac{1}{2} u_0 + \frac{D_t(x_1)}{2 D_x(x_1 -)}$$

on a fixed interior grid, where $D_x(x_1 -)$ denotes the one-sided derivative of D_x. If consistency of the integration scheme at x_1 is assumed, using (1.1), one obtains

$$\dot{x}_0 = u_1 + \frac{1}{2} u_0 - \frac{u_x D_1 + D_x u_1}{2 D_x}$$

$$= \frac{1}{2} (u_0 + u_1) - \frac{\Delta x_1 + \Delta x}{4} D_{xx} - \frac{\Delta x_1}{2} u_x(x_1) \qquad (4.5)$$

where D_{xx} approximates the difference quotient

$$\frac{D_x(x_1 +) - D_x(x_1 -)}{(\Delta x_1 + \Delta x)/2}$$

and Δx_1 denotes the grid spacing $x_1 - x_0$.

5. NUMERICAL RESULTS

The approach for the numerical solution of Eqs. (1.1) is to use a fixed grid for the interior and to employ Eqs. (4.2 - 4.4) to calculate the time-dependent positions of the boundary points as well as the velocity values. In order to have overall stability one couples a stable method for the interior grid points (e.g. upwind scheme/Lax-Wendroff scheme) with equations (4.2 - 4.4). One additional difficulty must be overcome. If the boundary point moves, the grid spacings Δx_1, Δx_{N+1} become very irregular with respect to the fixed grid spacing Δx, so the error will grow. In that case fixed grid points are activated or deactivated according to the rule that

$$\frac{\Delta x}{2} < \Delta x_1, \ \Delta x_{N+1} < \frac{3}{2} \Delta x.$$

Two model problems were calculated, which exhibit large movements of the boundary points:

1. The still water depth is defined by the parabola

   $$H(x) = \frac{1}{4} (1 - x^2)$$

   and the initial conditions are given by

   $$D(x,0) = \frac{1}{4} (x - 0.25) + H(x), \quad u(x,0) = 0.$$

   Comparisons of the numerical calculations with the exact solution [7] show reasonable results as depicted in Fig. 2. Small oscillations can be observed for interior grid points which are due to the finite-element method employed.

The location of the boundary points is to be calculated from this condition. Usually, extrapolation of water depth D is performed if a fixed grid is used. By Taylor expansion one gets the value of D(x) in terms of D_1, D_2, ... (respectively D_N, D_{N-1}, ...). Setting $D(x_b) = 0$ yields an implicit equation for the boundary point x_b [3, 4]. It is reported that second order extrapolation exhibits some kind of instabilities especially in cases where a bore near the boundary exists.

In contrast, when a moving grid is used interpolation is unnecessary. One simply drops the variational equation (2.3) for $\dot{h}$ for weight functions ϕ_0^t, ϕ_{N+1}^t and adds the natural conditions

$$D_1 = 0, \quad D_{N+1} = 0 . \tag{4.1}$$

Hence we demand $D \in X' := X \cap \{D(0) = D(N+1) = 0\}$.
From the duality principle it follows that $\alpha \neq 1$, because $(1-\alpha)\frac{\partial D}{\partial x}\phi_0 \notin X'$ must be assured. This condition prevents Eqs. (2.2 - 2.4) from becoming linear dependent and hence guarantees that $\dot{x}_0$ and $\dot{x}_{N+1}$ are uniquely determined.

The equations, valid at the boundaries, are directly derived from Eqs. (2.2 - 2.4) keeping in mind that integration is only performed over intervals $[x_0, x_1]$ and $[x_N, x_{N+1}]$, respectively. For i = 0 we find

$$<\dot{u} - u_x\dot{x} + u\frac{\partial u}{\partial x} + g\frac{\partial}{\partial x}(D-H),\ \phi_0^t> = 0 \tag{4.2}$$

$$\dot{D}_0 = 0 \tag{4.3}$$

$$(1-\alpha)<\dot{D} - D_x\dot{x} + u\frac{\partial D}{\partial x} + D\frac{\partial u}{\partial x}, \frac{\partial D}{\partial x}\phi_0^t> = 0 \tag{4.4}$$

where Eq. (4.3) replaces the variational equation for $\dot{D}$ and the first term in Eq. (2.4) vanishes because of Eq. (4.2). In a similar manner the equations at the boundary point x_{N+1} are found.

It is necessary that $\frac{\partial D}{\partial x}\Big|_{[x_0,x_1]} > 0$ $\left(\frac{\partial D}{\partial x}\Big|_{[x_N,x_{N+1}]} < 0\right)$ in order to have a positive water depth near the boundary. Solving for $\dot{x}_0$ results in

$$\dot{x}_0 = \frac{u_x<D,\phi_0> + D_x<u,\phi_0> + (\dot{D}_1 - D_x\dot{x}_1)<\phi_1,\phi_0>}{2D_x<\phi_1,\phi_0>}$$

$$= u_1 + \frac{1}{2}u_0 + \frac{\dot{D}_1 - D_x\dot{x}_1}{2D_x}$$

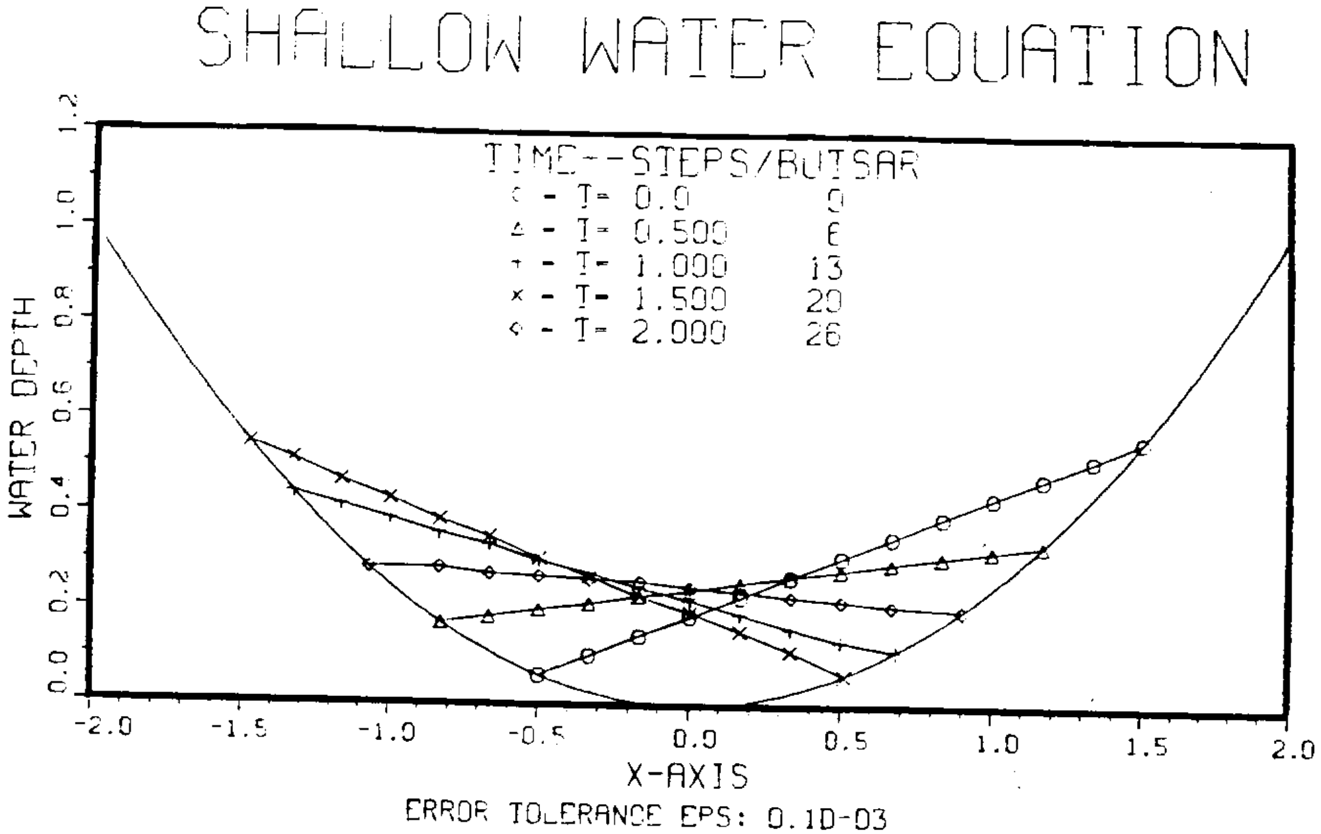

Fig. 2: Numerically calculated water depth at different times for problem 1. The analytic solution is a plane surface moving periodically to and fro.

2. For the second problem the still water depth is described by

$$H(x) = H_0 - \frac{3}{16}\left(1 - \frac{x^2}{8}\right)x^2 \qquad ; \; H(1) = 0$$

while the initial conditions are of the form

$$D(x,0) = \alpha \sin\left(K\frac{\pi}{2}(x+1)\right) + H(x)\,, \quad u(x,0) = 0.$$

Results are shown in Fig. 3.

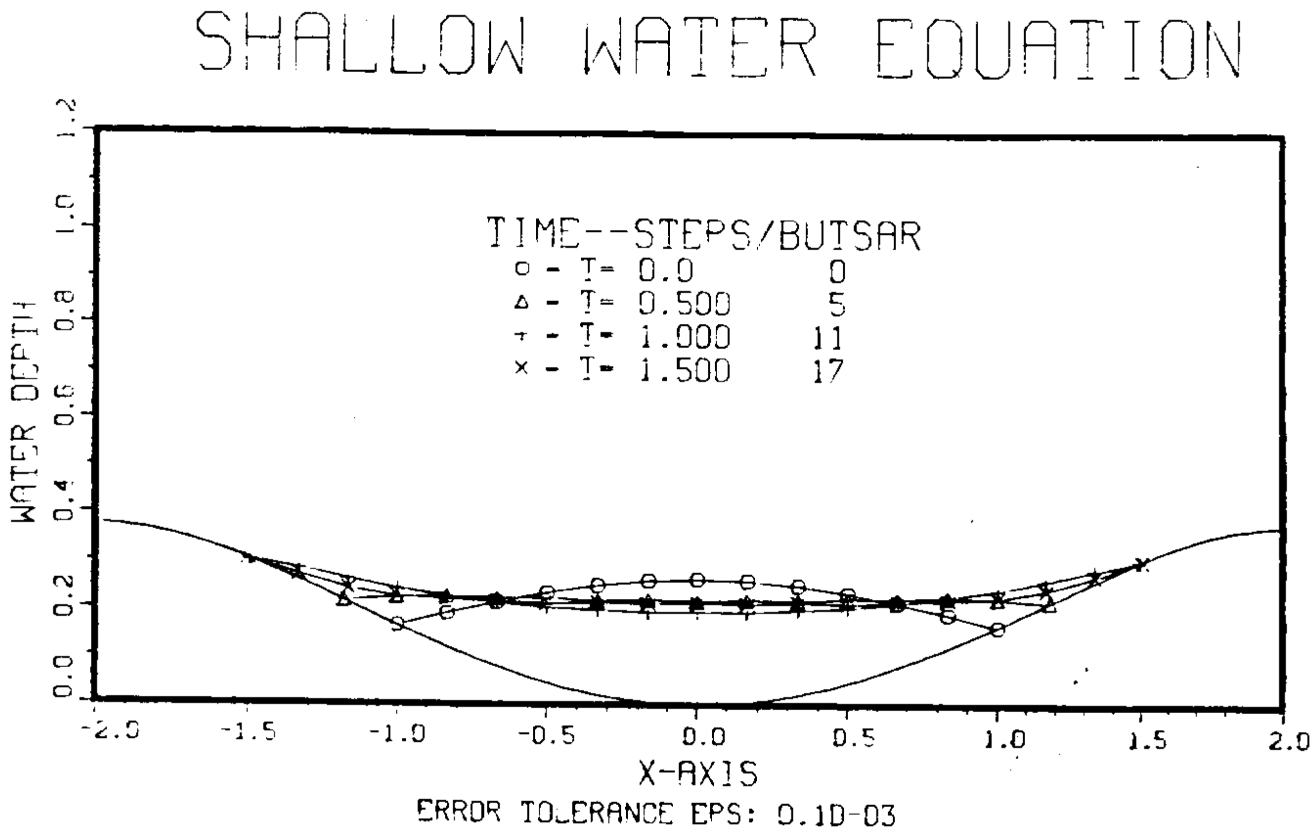

Fig. 3: Surface elevation at different time instants for problem 2.

6. CONCLUSIONS

General principles for the construction of an adaptive grid were used to calculate the time dependent domain together with the solution of the shallow-water equations. It is shown that a straight forward approach to insert the boundary conditions (in fact the definition of the boundary) fits perfectly in the framework of the grid generation method. In case of a smooth behavior of the solution in the interior, a fixed grid can be used. It is preferable to use classical numerical techniques for the fixed grid, which are much less time consuming compared with the fully adaptive scheme.

It should be pointed out that Eq. (4.5) demonstrates the limit of application of the scheme since the second term describes a viscous effect, which must be limited to get reasonable results. However, for rather small variations of both the surface elevation and the bottom line the calculations performed so far are encouraging to extend the method to two-dimensional cases.

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